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Preconditioned Methods for Solving the Incompressible and Low Speed Compressible Equations

ELI TURKEL*

Tel-Aviv University, Tel Aviv, Israel and
Institute for Computer Applications in Science and Engineering, NASA Langley Research Center, Hampton, Virginia 23665

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Acceleration methods are presented for solving the steady state incompressible equations. These systems are preconditioned by introducing artificial time derivatives which allow for a faster convergence to the steady state. We also consider the compressible equations in conservation form with slow flow. Two arbitrary functions α and β are introduced in the general preconditioning. An analysis of this system is presented and an optimal value for β is determined given a constant α. It is further sown that the resultant incompressible equations form a symmetric hyperbolic system and so are well posed. Several generalizations to the compressible equations are presented which extend previous results.

1. INTRODUCTION

In this study we consider ways of reaching a steady state for the incompressible fluid dynamics equations and also for low Mach number compressible flows. We shall only consider time-marching schemes that are represented by hyperbolic systems. Chorin [6] developed the artificial compressibility method which is further discussed by Peyret and Taylor [10]. We consider generalizations of this method by allowing artificial time derivatives in all the equations and not just the continuity equation. This allows for faster convergence and also facilitates a uniform treatment for both primitive variables and conservative variables. It is shown that the resultant equations form a symmetric hyperbolic system and so is well posed for both primitive and conservative formulations.

We next consider compressible flow with very low Mach numbers. As is well known, this system is stiff due to the large ratio of the acoustic and convective time scales. A number of people have considered preconditionings of these equations in various special cases, e.g., Viviand [20], Briley *et al.* [2], Choi and Merkle [4, 5], Rizzi [13], and Turkel [17, 19]. In this study we generalize these various approaches. In all cases we consider primitive variables p, u, v, plus an additional

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variable. After the analysis is complete it is shown how one can reformulate the system in conservation form.

As pointed out by Briley *et al.* [2], it is necessary to nondimensionalize the equations so that the pressure does not go to infinity as the Mach number goes to zero. In [2] this is accomplished by choosing $(\rho, \hat{c}_p, u, v, h_0)$ as the dependent variables. In this study we shall concentrate on (p, u, v) variables or else the conservation variables $(\rho, \rho u, \rho v, E)$. Instead, in the Runge-Kutta code [8] the variables used are nondimensionalized so that $p=\rho=1$ in the far field.

2. Incompressible Flow

In this section we consider the incompressible inviscid equations. Extensions to the viscous equations will be considered later while the following sections will discuss the effects of compressibility. We will only consider time-independent solutions. Nevertheless, since we shall discuss time-marching algorithms we begin with the time-dependent incompressible inviscid equations,

$$\begin{aligned} u_x + v_y &= 0 \\ u_t + uu_x + vu_y + p_x &= 0 \\ v_t + uv_x + vv_y + p_y &= 0. \end{aligned} \tag{2.1}$$

These equations can also be written in conservation form as

$$\begin{aligned} u_x + v_y &= 0 \\ u_t + (u^2 + p)_x + (uv)_y &= 0 \\ v_t + (uv)_x + (v^2 + p)_y &= 0. \end{aligned} \tag{2.2}$$

In this study we shall only consider smooth solutions to the systems (2.1) and (2.2). The only discontinuous solutions of interest are contact discontinuities, vortex sheets, etc., which are essentially linear phenomena and so are extensions of smooth flows. Shocked flows are not of interest for these equations and hence, the systems (2.1) and (2.2) are identical.

Since we are only interested in steady solutions we will modify the time derivatives that appear in (2.1) and (2.2). The simplest such modification is the pseudo-compressibility approach which adds a pressure time derivative to the continuity equation [6, 10, 15]. Then all the equations can be marched in time until a steady state is reached. We shall consider generalizations of this technique. All the time-dependent equations that we consider form hyperbolic systems. Since there is no decay mechanism except for boundaries we can accelerate to a steady state only by increasing the allowable time step. By normalizing the fastest speed it is shown in [19] that we accelerate the convergence when all the speeds are close together in absolute value. Conversely, the worst convergence occurs when the speeds of the system differ by orders of magnitude. It is also shown in [19] that in order to have a well-posed problem that is compatible with the steady state, especially in terms of boundary conditions, it is desirable to have a symmetric hyperbolic system. For a symmetric hyperbolic system, when the preconditioning matrix is positive definite we are guaranteed that we have not changed the appropriate number of boundary conditions and that we have not introduced any nonphysical time reversals.

We therefore consider the following extension of system (2.1)

$$\begin{aligned} \frac{1}{\beta^2} p_t + u_x + v_y &= 0 \\ \frac{\alpha u}{\beta^2} p_t + u_t + u u_x + v u_y + p_x &= 0 \\ \frac{\alpha v}{\beta^2} p_t + v_t + u v_x + v v_y + p_y &= 0. \end{aligned} \tag{2.3}$$

Here, α and β are functions to be determined. When, $\alpha = 0$ we recover the standard pseudo-compressibility method and we need only determine β. To form a conservation system we multiply the first equation by u and also v and then add to the second and third equations, respectively. The resulting system is

$$\begin{aligned} \frac{1}{\beta^2} p_t + u_x + v_y &= 0 \\ \frac{(\alpha+1)u}{\beta^2} p_t + u_t + (u^2 + p)_x + (uv)_y &= 0 \\ \frac{(\alpha+1)v}{\beta^2} p_t + v_t + (uv)_x + (v^2 + p)_y &= 0. \end{aligned} \tag{2.4}$$

Note 1. The system (2.4) is not truly conservative for time-dependent flows. However, we have in any case destroyed the time accuracy and the system is fully conservative in the steady state.

Note 2. Even for the original pseudo-compressibility approach, $\alpha = 0$, one should add pressure time derivatives to the momentum equations in the conservation form. Some authors, e.g., [4, 13] have not added these derivatives which amounts to choosing $\alpha = -1$.

Note 3. We shall do all the analysis on the nonconservative system (2.3). All the results will be equally valid for the conservative system (2.4) with the appropriate α and β.

We first rewrite (2.3) in vector form as

$$\begin{pmatrix} \dfrac{1}{\beta^2} & 0 & 0 \\ \dfrac{\alpha u}{\beta^2} & 1 & 0 \\ \dfrac{\alpha v}{\beta^2} & 0 & 1 \end{pmatrix} \begin{pmatrix} p \\ u \\ v \end{pmatrix}_t + \begin{pmatrix} 0 & 1 & 0 \\ 1 & u & 0 \\ 0 & 0 & u \end{pmatrix} \begin{pmatrix} p \\ u \\ v \end{pmatrix}_x + \begin{pmatrix} 0 & 0 & 1 \\ 0 & v & 0 \\ 1 & 0 & v \end{pmatrix} \begin{pmatrix} p \\ u \\ v \end{pmatrix}_y = 0 \tag{2.5}$$

or

$$E^{-1} w_t + A_0 w_x + B_0 w_y = 0 \tag{2.6}$$

with

$$w = (p, u, v)^t.$$

Multiplying (2.6) by E we rewrite (2.5) as

$$\begin{pmatrix} p \\ u \\ v \end{pmatrix}_t + \begin{pmatrix} 0 & \beta^2 & 0 \\ 1 & (1-\alpha)u & 0 \\ 0 & -\alpha v & u \end{pmatrix} \begin{pmatrix} p \\ u \\ v \end{pmatrix}_x + \begin{pmatrix} 0 & 0 & \beta^2 \\ 0 & v & -\alpha u \\ 1 & 0 & (1-\alpha) \end{pmatrix} \begin{pmatrix} p \\ u \\ v \end{pmatrix}_y = 0 \tag{2.7}$$

or

$$w_t + A w_x + B w_y = 0 \tag{2.8}$$

with

$$A = EA_0, \qquad B = EB_0.$$

In order to consider the wave speeds of (2.7) we Fourier transform the system. The wave speeds of (2.5) are given by the eigenvalues of

$$D = \omega_1 A + \omega_2 B, \qquad -1 \leqslant \omega_1,\ \omega_2 \leqslant 1, \tag{2.9}$$

where ω_1, ω_2 are the x and y components of the Fourier transform variable. Defining

$$q = u\omega_1 + v\omega_2, \tag{2.10}$$

we find that the eigenvalues of D are

$$d_0 = q, \tag{2.11a}$$

and

$$d_{\pm} = \tfrac{1}{2}[(1-\alpha)q \pm \sqrt{(1-\alpha)^2 q^2 + 4\beta^2}]. \tag{2.11b}$$

Remark. In the special case $\alpha = 1$, we have $d_{\pm} = \pm\beta$ and so the "acoustic" sound speed is isotropic and independent of the flow velocity.

We note that for all values of α and β, d_+ and d_- have opposite signs, i.e., $d_+ \cdot d_- = -\beta^2$ is always negative. This corresponds to subsonic flow for a compressible fluid which is appropriate for the incompressible case being considered.

We next consider the choice of β. We consider α as given and we wish to choose β to minimize the largest possible ratio of wave speeds. Thus, we wish to choose β so as to minimize $\max(|d_i/d_j|)$, $i, j = 0, \pm$. After some algebra we find that the appropriate β is given by

$$\frac{\beta^2}{q^2} = \begin{cases} 2-\alpha, & \alpha < 1, \text{ with condition no. } |2-\alpha| \\ \alpha, & \alpha \geqslant 1, \text{ with condition no. } \alpha. \end{cases} \tag{2.12}$$

Formula (2.12) is not useful since q, given by (2.10), is a function of the Foruier variable (ω_1, ω_2), while β must be given in physical space. Hence, we replace (2.12) by

$$\frac{\beta^2}{u^2+v^2} = \begin{cases} 2-\alpha, & \alpha < 1 \quad (2.13a) \\ \alpha, & \alpha \geqslant 1. \quad (2.13b) \end{cases}$$

The ratio of the fastest to the slowest speed now also depends on the ratio $(u^2+v^2)/q^2$ and will be larger than given in (2.12) unless $q^2 = u^2 + v^2$.

Remark. It follows from (2.12) that the optimal α is $\alpha = 1$ in which case the condition number is one, i.e., all the speeds have the same magnitude. Since β cannot be a function of the Fourier variables we must use (2.13) which means that the condition number is a function of ω_1/ω_2. Nevertheless, this is still the best result for a range of Fourier modes in multidimensions. We remind the reader that the original artificial compressibility corresponds to $\alpha = 0$ for the primitive variables and to $\alpha = -1$ for the conservative variables.

We note that in all these formulae β^2 is not constant but rather is a function of the speed, $u^2 + v^2$. To avoid difficulties near stagnation points (2.13) must be modified so that β cannot approach zero. For example, (2.13) can be changed to

$$\beta^2 = \begin{cases} \max[(2-\alpha)(u^2+v^2), \varepsilon] & \alpha < 1 \\ K \max[\alpha(u^2+v^2), \varepsilon] & \alpha \geqslant 1. \end{cases} \tag{2.14}$$

On dimensional grounds ε should be a fraction of $(u^2+v^2)_{\max}$. From later considerations, (2.16), K should be chosen slightly larger than one.

Until now we have only discussed the wave speeds, i.e., the eigenvalues of D, (2.9). We have shown that these eigenvalues are always real and so (2.5) is a hyperbolic system. We next wish to find out whether the system can be symmetrized. Gottlieb and Gustaffson [7] have suggested a general technique to check if a system can be simultaneously symmetrized. A necessary condition is that A and B can each be separately symmetrized. Since A can be symmetrized it can also be diagonalized. Furthermore, the diagonalization of A is unique except for exchanges of rows and columns and also an additional similarity transform using diagonal transformations. Thus after A has been diagonalized one only need check if B can be symmetrized by a diagonal similarity transform.

We first need the eigenvectors of A. This is also useful for constructing characteristic boundary conditions. If follows from (2.11) that

$$a_0 = u$$
$$a_\pm = \frac{(1-\alpha)u \pm \sqrt{(1-\alpha)^2 u^2 + 4\beta^2}}{2} \tag{2.15}$$

are the eigenvalues of A in (2.8). Let

$$T = \begin{pmatrix} \dfrac{1}{a_+ - a_1} & \dfrac{a_+}{a_+ - a_-} & 0 \\ \dfrac{-1}{a_+ - a_-} & \dfrac{-a_-}{a_+ - a_-} & 0 \\ \dfrac{-\alpha v}{(u-a_+)(u-a_-)} & \dfrac{-\alpha u v}{(u-a_+)(u-a_-)} & 1 \end{pmatrix}, \quad T^{-1} = \begin{pmatrix} -a_- & -a_+ & 0 \\ 1 & 1 & 0 \\ \dfrac{\alpha v}{u-a_+} & \dfrac{\alpha v}{u-a_-} & 1 \end{pmatrix}.$$

Then $\det(T^{-1}) = \sqrt{(1-\alpha)^2 y^2 + 4\beta^2} \neq 0$ and so the transformation is nonsingular. Furthermore, the columns of T^{-1} are the eigenvectors of A. It then follows that

$$TAT^{-1} = \begin{pmatrix} a_+ & 0 & 0 \\ 0 & a_- & 0 \\ 0 & 0 & u \end{pmatrix}$$

and

$$TBT^{-1} = \begin{pmatrix} \dfrac{(1-\alpha)a_+ v}{a_+ - a_-} & \dfrac{(1+\alpha)a_+^2 v}{(a_+ - a_-)(u-a_-)} & \dfrac{-a_+(u-a_+)}{a_+ - a_-} \\ \dfrac{-(1+\alpha)a_-^2 v}{(a_+ - a_-)(u-a_+)} & \dfrac{-(1-\alpha)a_- v}{a_+ - a_-} & \dfrac{a_-(u-a_-)}{a_+ - a_-} \\ -a_- - \dfrac{\alpha v a_-}{(u-a_+)(u-a_-)} & -a_+ - \dfrac{\alpha v a_+}{(u-a_+)(u-a_-)} & v \end{pmatrix}.$$

Let $D_0 = \operatorname{diag}(d_1, d_2, d_3)$ with

$$d_1 = a_-\sqrt{(a_+ - a_-)(u-a_-)}, \qquad d_2 = a_+\sqrt{(a_{++} - a_-)(a_+ - u)},$$
$$d_3 = (a_+ - u)(a_- - u)\sqrt{-a_+ a_- / (\beta^2 - \alpha(u^2 + v^2))}.$$

Then $D_0 TAT^{-1} D_1^{-1}$ is still diagonal, while $D_0 TBT^{-1} D_0^{-1}$ is now symmetric. It follows from the definition of a_+, a_- that $a_+ > 0 > a_-$. It can then be shown that D_0 is real and hence the system is symmetrizable if and only if $q^2 \leqslant d_+^2$ for all (ω_1, ω_2) or, equivalently,

$$\beta^2 > \alpha(u^2 + v^2). \tag{2.16}$$

We note that from (2.13) we have, that the optimal β, for $\alpha \geqslant 1$, is gotten by choosing an equality in (2.16) rather than an inequality. Hence, if one wishes the system to be both close to optimal and symmetrizable we should choose β^2 slightly larger than $u^2 + v^2$. Furthermore, for $\alpha < 1$, (2.13a) implies (2.16) automatically. For $\alpha \leqslant 0$, (2.16) is always satisfied for all β.

When using an explicit method we need an upper bound on the largest eigenvalue of D. A typical explicit scheme has a stability criterion of the form

$$\frac{\Delta t}{\Delta} \leqslant \frac{K}{d_+}, \tag{2.17}$$

where Δ is a typical mesh length and K is a constant that depends on the scheme. Using (2.11b) we replace d_+ in (2.17) by its upper bound

$$d_+ \leqslant \frac{(u^2 + v^2)}{2} \left[(1 - \alpha + \sqrt{(1-\alpha)^2 + 4\beta^2/(u^2 + v^2)}\right]$$

with $\beta^2/(u^2 + v^2)$ given by (2.14). If we use a general curvilinear mesh then the corresponding formula is given by (3.20), with $c = \infty$.

The previous discussion has been scheme-independent and relied only on equalizing the wave speeds for the differential equation. We now discuss the implementation for some difference schemes. For an explicit scheme the time step is restricted by the fastest moving wave. Thus, the previous analysis insists that the time step chosen by a stability analysis should not be inappropriate for the slower waves. If the wave speeds differ significantly then the slower waves will propagate very slowly and convergence will also be slow. Furthermore, for most explicit schemes the damping of the scheme is small for small Δt and so the slowly moving waves will not be damped very much. Hence, our analysis is certainly appropriate for standard explicit schemes.

Using an implicit method it is less clear that the stiffness of the system matters. If one uses a backward Euler method then one can show [8] that for larger Δt that one approaches the classical Newton–Raphson iteration scheme. In this case the convergence is not very affected by the stiffness of the system. In [4] computations are presented that show fast convergence for one-dimensional problems. However, in multidimensions it is not practical to invert the matrix that one gets using a fully implicit scheme. Instead one frequently uses an ADI-type algorithm. In this case one should not choose very large Δt [16], but rather one close to the explicit Courant condition. This occures because of the $(\Delta t)^2 AB$ term that is created by the splitting. Hence, again a Δt that is appropriate for the fast waves is inappropriate for the slow waves. Hence, our preconditioning which will equilibrate the wave speeds will also accelerate ADI-type methods. Using the notation of Beam and Warming [1], we write an implicit scheme for (2.4) as

$$\left[E^{-1}+\Delta t\left(\frac{\partial}{\partial x}A_0^n+\frac{\partial}{\partial y}B_0^n\right)\right]\Delta w=-\Delta t(f_x^n+g_y^n), \tag{2.17}$$

where $\Delta w=w^{n+1}-w^n$,

$$f=\begin{pmatrix} u \\ u^2+p \\ uv \end{pmatrix}, \qquad g=\begin{pmatrix} u \\ uv \\ v^2+p \end{pmatrix}$$

$$A_0=\frac{\partial f}{\partial w}=\begin{pmatrix} 0 & 1 & 0 \\ 1 & 2u & 0 \\ 0 & v & u \end{pmatrix}, \qquad B_0=\frac{\partial g}{\partial w}=\begin{pmatrix} 0 & 0 & 1 \\ 0 & v & u \\ 1 & 0 & 2v \end{pmatrix}$$

$$E^{-1}=\begin{pmatrix} \frac{1}{\beta^2} & 0 & 0 \\ \frac{(\alpha+1)u}{\beta^2} & 1 & 0 \\ \frac{(\alpha+1)u}{\beta^2} & 0 & 1 \end{pmatrix}, \qquad E=\begin{pmatrix} \beta^2 & 0 & 0 \\ -(\alpha+1)u & 1 & 0 \\ -(\alpha+1)v & 0 & 1 \end{pmatrix}.$$

We rewrite (2.17) as

$$E^{-1}\left[I+\Delta t\left(E^n\frac{\partial}{\partial x}A_0^n+E^n\frac{\partial}{\partial y}B_0^n\right)\right]\Delta w=-\Delta t(f_x^n+g_y^n). \tag{2.18}$$

We now apply an approximate factorization to (2.18) and ignore errors in the conservation form of the left-hand side to get

$$E^{-1}\left[I+\Delta t\frac{\partial}{\partial x}EA_0\right]\left[I+\Delta t\frac{\partial}{\partial y}EB_0\right]\Delta w=-\Delta t(f_x^n+g_y^n). \tag{2.19}$$

Since the matrices E, $A=EA_0$, and $B=EB_0$ are well conditioned there is no way that the splitting error can slow down the convergence compared with the standard ADI splitting.

For $\alpha=0$ we need only invert 2×2 blocks. For general α we can use the factorization suggested by Pulliam and Steger [11]. Hence,

$$A=EA_0=\begin{pmatrix} 0 & \beta^2 & 0 \\ 1 & (1-\alpha)u & 0 \\ 0 & -\alpha v & u \end{pmatrix}, \qquad B=EB_0=\begin{pmatrix} 0 & 0 & \beta^2 \\ 0 & v & -\alpha u \\ 1 & 0 & (1-\alpha)v \end{pmatrix}$$

can be diagonalized, i.e., $UAU^{-1}=D_1$ and $VAV^{-1}=D_2$. Ignoring, again, conservation errors in the left-hand side of (2.19), we rewrite (2.19) as

$$E^{-1}U\left[I+\Delta t\frac{\partial}{\partial x}D_1\right]U^{-1}V\left[I+\Delta t\frac{\partial}{\partial y}D_2\right]V^{-1}\,\Delta w=-\Delta t(f_x^n+g_y^n) \quad (2.20)$$

and so we need only invert scalar tridiagonal matrices rather than block tridiagonal matrices.

In practice one usually solves the viscous equations rather than the inviscid equations. The easiest remedy is simply to add the viscous terms to the right-hand side of (2.19). One usually finds, for large Reynolds numbers, that the time step is restricted only by the inviscid terms. Hence, there is no need to include a viscous Jacobian on the left-hand side of (2.19). Furthermore, the preconditioner, E, still equilibrates the inviscid time steps and reduces the splitting error in (2.19).

We next consider the implementation of the scheme on a staggered mesh (see Fig. 1). The steady state equations are independent of Δt and so we retain the improved accuracy of the staggered grid independent of our treatment of the time-marching algorithm. Thus, for example, we discretize the x-momentum equation in (2.3) by

$$\frac{\alpha u_{i+1/2,j}}{2\beta^2_{i+1/2,j}}\frac{(p^{n+1}_{i+1,j}-p^n_{i+1,j}+p^{n+1}_{i,j}-p^n_{i,j})}{\Delta t}+\frac{u^{n+1}_{i+1/2,j}-u^{n+1}_{i+1/2,j}}{\Delta t}$$
$$+\text{ usual space differentiation}=0, \quad (2.21)$$

where

$$\beta^2_{i+1/2,j}=K_1\left[u^2_{i+1/2,j}+\left(\frac{v_{i,j+1/2}+v_{i+1,j+1/2}+v_{i,j-1/2}+v_{i+1,j-1/2}}{4}\right)\right]^2$$

and K_1 is a function of α given in (2.14). Using an explicit scheme p_t at (i,j) is already known from the first equation. With an implicit scheme we now have contributions of p_t in the momentum equations which contribute to both the diagonal and off-diagonal blocks.

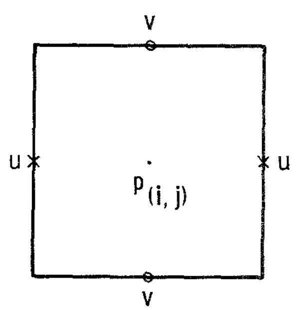

FIGURE 1

In this section we have only considered Cartesian coordinates. The extension to general curvilinear coordinates is straightforward. This will be done in the sections on compressible flow which will contain the incompressible case as a limiting solution. Here we shall only show that the matrices are simultaneously symmetrizable in curvilinear coordinates whenever they are symmetrizable in Cartesian coordinates. Consider the equation in Cartesian coordinates (X, Y),

$$w_t + Aw_X + Bw_Y = 0. \tag{2.22}$$

Let $x = x(X, Y)$, $y = y(X, Y)$ be general coordinates then

$$w_t + A_1 w_x + B_1 w_y = 0 \tag{2.23}$$

with

$$A_1 = Ax_X + Bx_Y, \qquad B_1 = Ay_X + By_Y. \tag{2.24}$$

Since A_1 and B_1 are linear combination of A and B it follows that whenever A and B can be symmetrized simultaneously so can A_1 and B_1.

3. Compressible (p, u, v, S) System

In the previous section we have considered incompressible flow where the unknowns are (p, u, v). We next consider the compressible equations concentrating on low speed flow. Since our analysis is local we need only consider flows that locally have a small Mach number. The flow can even be supersonic in other regions. Hence, it is useful to consider the conservation form of the equations. In considering the compressible equations we need an additional unknown. Three possibilities are entropy S, or density ρ, or else to use Bernoulli's law stating that the total enthalpy is constant, i.e., isoenergetic flow. In all cases we shall ultimately cast the equations in conservation form but the three possibilities lead to different preconditioning. As before we shall do the analysis on the primitive variables and only at the end shall we derive the conservation variable version. In this section we [illegible]

The time-dependent Euler equations can be written in Cartesian coordinates (X, Y) as

$$\begin{aligned} \frac{1}{\rho c^2} p_t + \frac{1}{\rho c^2}(up_X + vp_Y) + u_X + v_Y &= 0 \\ u_t + uu_X + vu_Y + p_X/\rho &= 0 \\ v_t + uv_X + vv_Y + p_Y/\rho &= 0 \\ S_t + uS_X + vS_Y &= 0, \end{aligned} \tag{3.1}$$

where

$$\rho = \rho(p, S).$$

We now introduce curvilinear coordinates $x = x(X, Y)$, $y = y(X, Y)$. The Euler equations in (x, y) coordinates are

$$\begin{aligned}
\frac{J}{\rho c^2} p_t + \frac{1}{\rho c^2}(Up_x + Vp_y) + u_x Y_y - v_x X_y - u_y Y_x + v_y X_x &= 0 \\
Ju_t + Uu_x + Vv_y + (p_x Y_y - p_y Y_x)/\rho &= 0 \\
Jv_t + Uv_x + Vv_y + (-p_x X_y + p_y X_x)/\rho &= 0 \\
JS_t + US_x + VS_y &= 0,
\end{aligned} \tag{3.2}$$

where $\rho = \rho(p, S)$, and

$$U = uY_y - vX_y, \qquad V = -uY_x + vX_x, \qquad J = X_x Y_y - X_y Y_x. \tag{3.2a}$$

We precondition this system by a generalization of (2.5). We thus obtain

$$J\begin{pmatrix} \frac{1}{\rho\beta^2} & 0 & 0 & 0 \\ \frac{\alpha u}{\rho\beta^2} & 1 & 0 & 0 \\ \frac{\alpha v}{\rho\beta^2} & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} p \\ u \\ v \\ S \end{pmatrix}_t + \begin{pmatrix} \frac{U}{\rho c^2} & Y_y & -X_y & 0 \\ \frac{Y_y}{\rho} & U & 0 & 0 \\ -X_y/\rho & 0 & U & 0 \\ 0 & 0 & 0 & U \end{pmatrix}\begin{pmatrix} p \\ u \\ v \\ S \end{pmatrix}_x$$

$$+\begin{pmatrix} \frac{V}{\rho\beta^2} & -Y_x & X_x & 0 \\ -Y_x/\rho & V & 0 & 0 \\ \frac{X_x}{\rho} & 0 & V & 0 \\ 0 & 0 & 0 & V \end{pmatrix}\begin{pmatrix} p \\ u \\ v \\ S \end{pmatrix}_y = 0. \tag{3.3}$$

Since the entropy decouples on the matrix level (we freeze coefficients and then ρ no longer depends on S), for stability theory we can reduce (3.3) to the simpler equivalent form

$$J\begin{pmatrix} \frac{1}{\rho\beta^2} & 0 & 0 \\ \frac{\alpha u}{\rho\beta^2} & 1 & 0 \\ \frac{\alpha v}{\rho\beta^2} & 0 & 1 \end{pmatrix}\begin{pmatrix} p \\ u \\ v \end{pmatrix}_t + \begin{pmatrix} \frac{U}{\rho c^2} & Y_y & -X_y \\ Y_y/\rho & U & 0 \\ -X_y/\rho & 0 & U \end{pmatrix}\begin{pmatrix} p \\ u \\ v \end{pmatrix}_x$$

$$+\begin{pmatrix} \frac{V}{\rho c^2} & -Y_x & X_x \\ -Y_x/\rho & V & 0 \\ \frac{X_x}{\rho} & 0 & V \end{pmatrix}\begin{pmatrix} p \\ u \\ v \end{pmatrix}_y = 0. \tag{3.4}$$

We note that (3.4) is very similar to (2.5). In fact, setting $c = \beta$ and $\rho = 1$ and using Cartesian coordinates, (3.4) reduces to (2.5). As before, we rewrite (3.4) as

$$JE^{-1}w_t + A_0 w_x + B_0 w_y = 0. \tag{3.5}$$

Multiplying (3.4) by E, we obtain

$$Jw_t + Aw_x + Bw_y = 0 \tag{3.6}$$

with $w = (p, u, v)^t$ and

$$A = EA_0 = \begin{pmatrix} \frac{\beta^2 U}{c^2} & \rho\beta^2 Y_y & -\rho\beta^2 X_y \\ \frac{-\alpha u U}{\rho c^2} + \frac{Y_y}{\rho} & -\alpha u Y_y + U & \alpha u X_y \\ \frac{-\alpha v U}{\rho c^2} - \frac{X_y}{\rho} & -\alpha v Y_y & \alpha v X_y + U \end{pmatrix},$$

$$B = EB_0 = \begin{pmatrix} \frac{\beta^2 V}{c^2} & -\rho\beta^2 Y_x & \rho\beta^2 X_x \\ \frac{-\alpha u V}{\rho c^2} - \frac{Y_x}{\rho} & \alpha u Y_x + V & -\alpha u X_x \\ \frac{-\alpha v V}{\rho c^2} + \frac{X_x}{\rho} & \alpha v Y_x & -\alpha v X_x + V \end{pmatrix}, \tag{3.7}$$

and the Jacobian J is given by (3.2a). To find the wave speeds we again examine the eigenvalues of

$$D = \omega_1 A + \omega_2 B, \qquad -1 \leqslant \omega_1,\ \omega_2 \leqslant 1. \tag{3.8}$$

We define

$$l_1 = Y_y\omega_1 - Y_x\omega_2, \qquad l_2 = -X_y\omega_1 + X_x\omega_2$$

and (3.9)

$$q = U\omega_1 + V\omega_2 = ul_1 + vl_2;$$

U and V were defined in (3.2a). Then the eigenvalues of D are

$$d_0 = q \tag{3.10a}$$

and

$$d_\pm = \frac{1}{2}\left[\left(1-\alpha+\frac{\beta^2}{c^2}\right)q \pm \sqrt{\left(1-\alpha+\frac{\beta^2}{c^2}\right)^2 q_2 + 4\left(l_1^2+l_2^2-\frac{q^2}{c^2}\right)\beta^2}\right]. \tag{3.10b}$$

We note that without preconditioning we have

$$d_\pm = q \pm \sqrt{l_1^2+l_2^2}\,c.$$

Remark. If we consider the special case $\alpha = 1 + \beta^2/c^2 \simeq 1$, then $d_\pm = \pm\beta\sqrt{l_1^2+l_2^2-q^2/c^2} \simeq \pm\beta\sqrt{l_1^2+l_2^2}$ for low speed flow. Hence, the acoustic waves moves with a speed independent of the velocities u and v and this wave is isotropic except for grid effects. Also, we note that (3.10) is independent of c.

We also know that

$$\begin{aligned} l_1^2+l_2^2 &= (X_y^2+Y_y^2)\omega_1^2 + (X_x^2+Y_x^2)\omega_2^2 - 2(X_xX_y+Y_xY_y)\omega_1\omega_2 \leqslant L^2 \\ &= [X_x^2+Y_x^2+X_y^2+Y_y^2+2|X_xX_y+Y_xY_y|]. \end{aligned} \tag{3.11}$$

For subsonic flow d_+ and d_- have the opposite signs. In fact

$$d_+d_- = -(l_1^2+l_2^2-q^2/c^2)\beta^2 < 0$$

whenever $u^2+v^2<c^2$. For an orthogonal mesh $X_xX_y+Y_xY_y=0$ and so the expression for L^2, (3.11) simplifies.

We next consider the choice for β. We wish to choose β so as to minimize the largest possible ratio of the wave speeds, i.e., to minimize the maximum of the ratio of the d's in (3.10). In order to simplify the algebra we assume that $q^2/c^2 \ll 1$, i.e., slow flow, we also assume that $\beta^2/c^2 \ll 1$. The optimal β is the computed as (cf. (2.12)),

$$\frac{\beta^2(l_1^2+l_2^2)}{q^2} = \begin{cases} 2-\alpha+O(q^2/c^2) & \alpha<1 \\ \alpha+O(q^2/c^2) & \alpha\geqslant 1. \end{cases} \tag{3.12}$$

The condition number is the same as (2.12) to within $O((\beta^2+q^2)/c^2)$. Similar to the incompressible case, (3.12) is not of immediate use since l_1, l_2, and q all depend on the Fourier variable (ω_1, ω_2), see (3.9). Instead, we suggest using

$$\frac{\beta^2 L^2}{u^2+v^2} = \begin{cases} 2-\alpha & \alpha<1 \\ \alpha & \alpha \geqslant 1, \end{cases} \tag{3.13}$$

where L is given in (3.11).

We next rewrite (3.3) in terms of the conservation variables $(\rho, \rho u, \rho v, E)$ with $E = p/(\gamma-1) + (\rho/2)(u^2+v^2)$. We then obtain

$$J \begin{pmatrix} \dfrac{z_1}{\gamma-1} p_t + \rho_t \\ \dfrac{u z_2}{\gamma-1} p_t + (\rho u)_t \\ \dfrac{v z_2}{\gamma-1} p_t + (\rho v)_t \\ \dfrac{z_3}{\gamma-1} p_t + E_t \end{pmatrix} + F_x + G_y = 0, \tag{3.14}$$

where F and G are the standard Euler fluxes in curvilinear coordinates, J is the Jacobian, and

$$\begin{aligned} z_1 &= (\gamma-1)\left(\frac{1}{\beta^2} - \frac{1}{c^2}\right) \\ z_2 &= (\gamma-1)\left(\frac{\alpha+1}{\beta^2} - \frac{1}{c^2}\right) = z_1 + \frac{\alpha(\gamma-1)}{\beta^2} \\ z_3 &= (\gamma-1)\frac{1}{\gamma-1}\left(\frac{c^2}{\beta^2} - 1\right) + \left(\frac{2\alpha+1}{\beta^2} - \frac{1}{c^2}\right)\left(\frac{u^2+v^2}{2}\right) \\ &= z_1 h + (\gamma-1)\,\alpha(u^2+v^2)/\beta^2, \qquad h = \frac{c^2}{\gamma-1} + \frac{u^2+v^2}{2}. \end{aligned} \tag{3.15}$$

Thus, as expected, we recover the correct steady state equations. We can also eliminare p_t in (3.14) and obtain equations only in terms of ρ_t, $(\rho u)_t$, $(\rho v)_t$, and E_t. We then have

$$J(I+Q) \begin{pmatrix} \rho \\ \rho u \\ \rho v \\ E \end{pmatrix}_t + F_x + G_y = 0, \tag{3.16}$$

where I is the identity matrix and

$$Q=\begin{pmatrix} R^2z_1 & -uz_1 & -vz_1 & z_1 \\ R^2z_2u & -u^2z_2 & -uvz_2 & z_2u \\ R^2z_2v & -uvz_2 & -v^2z_2 & z_2v \\ R^2z_3 & -uz^3 & -vz_3 & z_3 \end{pmatrix} \tag{3.17}$$

and $R^2=(u^2+v^2)/2$. When $\alpha=0$ and $\beta^2=u^2+v^2$ this reduces to the preconditioner found in [18] by a different technique. We note that for $\alpha=0$ the optimal β given by (3.13) is $\beta^2=2(u^2+v^2)/L^2$. Furthermore, we can invert $I+Q$ simply to get

$$(I+Q)^{-1}=I-\frac{\beta^2}{c^2}Q. \tag{3.18}$$

Because of the structure of Q we can multiply Q times a vector using seven multiplications.

As before, the stability criterion for a typical explicit scheme for (3.16) has the form

$$\frac{\Delta t}{J}\leqslant K/d_+, \tag{3.19}$$

where K is a constant that depends on the scheme. It follows from (3.9)–(3.11) that

$$\frac{\Delta t}{J}\leqslant \frac{2K}{(1-\alpha+(\beta^2/c^2)\sqrt{U^2+V^2}+\sqrt{(1-\alpha+(\beta^2/c^2))^2(U^2+V^2)+4(L^2-(U^2+V^2)/c^2)\beta^2}}, \tag{3.20}$$

where L is defined in (3.11), is a sufficient condition for stability. For slow speed flow we can ignore all terms of the order $(U^2+V^2)/c^2$ and β^2/c^2. Also since $\beta^2=O(u^2+v^2)$ by (3.13), we see that Δt is independent of c and depends only on the local velocity. As pointed out previously the special choice $\alpha=1+\beta^2/c^2$ simplifies the formulas. We then find that

$$\frac{\Delta t}{J}\leqslant\frac{K}{\beta\sqrt{L^2-(U^2+V^2)/c^2}} \quad \text{for} \quad \alpha=1+\beta^2/c^2. \tag{3.21}$$

As long as the flow is subsonic the square root is meaningful.

As is the incompressible case we find that the matrices A and B can be simultaneously symmetrized when

$$\beta^2>\alpha(u^2+v^2). \tag{3.22}$$

In forming the preconditioned system (3.16) we eliminated the pressure term p_t from (3.14). Since we are not interested in the time-dependent solution we can instead eliminate ρ_t,

$$\left(\frac{u^2+v^2}{2}\right)\rho_t=\frac{p_t}{\gamma-1}+u(\rho u)_t+v(\rho v)_t-E_t. \tag{3.23}$$

As before, we need to do something spcial in the neighborhood of stagnation points. This system now solves for $(p, \rho u, \rho v, E)$ and so is more similar to the incompressible limit.

4. Compressible (p, u, v, ρ) System

In the previous section we appended the entropy equation to the incompressible (p, u, v) equations and did not precondition the S equation. This had the benefit that the entropy equation decoupled and so even in the compressible case we needed to only consider three equations (see (3.2)–(3.4)). Choi and Merkle [4, 5] have discussed a (p, u, v, ρ) formulation which we now analyze in more detail.

Again considering curvilinear coordinates $x = x(X, Y)$ and $y = y(X, Y)$, the Euler equations are (compare with (3.2)):

$$\begin{aligned}\frac{J}{\rho c^2} p_t + \frac{1}{\rho c^2}(Up_x + Vp_y) + u_x Y_y - v_x X_y - u_y Y_x + v_y X_x &= 0\\ Ju_t + Uu_x + Vv_y + (p_x Y_y - p_y Y_x)/\rho &= 0\\ Jv_t + Uv_x + Vv_y + (-p_x X_y + p_y X_x)/\rho &= 0\\ J\rho_t + U\rho_x + V\rho_y + \rho(u_x Y_y - v_x X_y - u_y Y_x + v_y X_x) &= 0,\end{aligned} \tag{4.1}$$

where

$$U = uY_y - vX_y, \qquad V = -uY_x + vX_x, \qquad J = X_x Y_y - X_y Y_x.$$

We precondition this system similar to (3.3), where again the last equation is not preconditioning. Thus, we are now not changing the ρ equation rather than the S equation of (3.3). We then obtain

$$J\begin{pmatrix}\frac{1}{\rho\beta^2} & 0 & 0 & 0\\ \frac{\alpha u}{\rho\beta^2} & 1 & 0 & 0\\ \frac{\alpha v}{\rho\beta^2} & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{pmatrix}\begin{pmatrix}p\\ u\\ v\\ \rho\end{pmatrix}_t + \begin{pmatrix}\frac{U}{\rho c^2} & Y_y & -X_y & 0\\ \frac{X_y}{\rho} & U & 0 & 0\\ -X_y/\rho & 0 & U & 0\\ 0 & \rho Y_y & -\rho X_y & U\end{pmatrix}\begin{pmatrix}p\\ u\\ v\\ \rho\end{pmatrix}_Y \tag{4.2}$$

$$+\begin{pmatrix}\frac{V}{\rho c^2} & -Y_x & x_x & 0\\ -Y_x/\rho & V & 0 & 0\\ X_x/\rho & 0 & V & 0\\ 0 & -\rho Y_x & \rho X_x & V\end{pmatrix}\begin{pmatrix}p\\ u\\ v\\ \rho\end{pmatrix}_y = 0.$$

In order to facilitate comparisons with (3.3) we change variables in (4.2) to a (p, u, v, S) system. Formally, we define $S = \ln[p/p^\gamma]$ and so $d\rho = (1/c^2)(dp - p\,dS)$. Substituting into (4.2) we get

$$
J\begin{pmatrix} \dfrac{1}{\rho\beta^2} & 0 & 0 & 0 \\ \dfrac{\alpha u}{\rho\beta^2} & 1 & 0 & 0 \\ \dfrac{\alpha v}{\rho\beta^2} & 0 & 1 & 0 \\ \dfrac{1}{p}\left(\dfrac{c^2-1}{\beta^2}\right) & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} p \\ u \\ v \\ S \end{pmatrix}_t + \begin{pmatrix} \dfrac{U}{\rho c^2} & Y_y & -X_y & 0 \\ Y_y/\rho & U & 0 & 0 \\ -X_y/\rho & 0 & U & 0 \\ 0 & 0 & 0 & U \end{pmatrix}\begin{pmatrix} p \\ u \\ v \\ S \end{pmatrix}_x
$$

$$
+ \begin{pmatrix} \dfrac{V}{\rho c^2} & -Y_x & X_x & 0 \\ -Y_x/\rho & V & 0 & 0 \\ X_x/\rho & 0 & V & 0 \\ 0 & 0 & 0 & V \end{pmatrix}\begin{pmatrix} p \\ u \\ v \\ S \end{pmatrix}_y = 0. \tag{4.3}
$$

Comparing (3.3) with (4.3) we see that using the (p, u, v, ρ) formulation has introduced an additional p_t preconditioning into the entropy equation. Hence, the entropy equation no longer decouples from the previous three equations. This complicates the analysis. The advantage of the (p, u, v, ρ) is that it simplifies the preconditioning in conservative variables as will be seen later. Solving for $(p, u, v, S)_t$ we find

$$
J\begin{pmatrix} p \\ u \\ v \\ S \end{pmatrix}_t + \begin{pmatrix} \dfrac{\beta^2}{c^2}U & \rho\beta^2 Y_y & -\rho\beta^2 X_y & 0 \\ \dfrac{-\alpha u U}{\rho c^2} + Y_y/\rho & -\alpha u Y_y + U & -\alpha u X_y & 0 \\ \dfrac{-\alpha v U}{\rho c^2} - Y_y/\rho & -\alpha v Y_y & -\alpha v X_y & 0 \\ \dfrac{-\gamma U}{\rho c^2}\left(1 - \dfrac{\beta^2}{c^2}\right) & -\gamma Y_y\left(\dfrac{1-\beta^2}{c^2}\right) & -\gamma X_y\left(\dfrac{1-\beta^2}{c^2}\right) & U \end{pmatrix}\begin{pmatrix} p \\ u \\ v \\ S \end{pmatrix}_x
$$

$$
+\begin{pmatrix} \frac{\beta^2}{c^2}V & -\rho\beta^2 Y_x & \rho\beta^2 X_x & 0 \\ \frac{-\alpha uV}{\rho c^2}-\frac{Y_x}{\rho} & \alpha u Y_x + V & -\alpha u X_x & 0 \\ \frac{-\alpha v}{\rho c^2}+\frac{X_x}{\rho} & -\alpha v Y_x & -\alpha v X_x + V & 0 \\ \frac{-\alpha V}{\rho c^2}\left(1-\frac{\beta^2}{c^2}\right) & -\alpha Y_x\left(1-\frac{\beta^2}{c^2}\right) & \gamma X_x\left(1-\frac{\beta^2}{c^2}\right) & V \end{pmatrix} \begin{pmatrix} p \\ u \\ v \\ S \end{pmatrix}_y = 0. \tag{4.4}
$$

or

$$
Jw_t + Aw_x + Bw_y. \tag{4.5}
$$

Comparing the matrices A and B of (3.6)–(3.7) with that of (4.4)–(4.5) the eigenvalues of any linear combination of A and B are unchanged since the last column has zeros except for the corner element. Hence, all the preconditionings that were considered before in Section 3 are equally efficient for the (p, u, v, ρ) system. In particular, an optimal β is given by (3.13) and the time step restriction for a typical explicit method is given by (3.20) and (3.21).

We now rewrite our preconditioned (p, u, v, ρ) system (4.2) in terms of conservation variables. This becomes

$$
\begin{pmatrix} \rho_t \\ \frac{uz_2}{\gamma-1}p_t + (\rho u)_t \\ \frac{vz_2}{\gamma-1}p_t + (\rho v)_t \\ \frac{z_3}{\gamma-1}p_t + E_t \end{pmatrix} + F_x + G_y = 0 \tag{4.6}
$$

with

$$
\begin{aligned} z_2 &= (\gamma-1)\alpha/\beta^2 \\ z_3 &= (\gamma-1)\left[\frac{1}{\gamma-1}\left(\frac{c^2}{\beta^2}-1\right)+\frac{\alpha(u^2+v^2)}{\beta^2}\right] \end{aligned} \tag{4.7}
$$

(compare (3.15)). Eliminating p_t in (4.6) we find that

$$
J(I+Q)\begin{bmatrix} \rho \\ \rho u \\ \rho v \\ E \end{bmatrix}_t + F_x + G_y = 0, \tag{4.8}
$$

where I is the identity matrix and

$$Q=\begin{pmatrix} 0 & 0 & 0 & 0 \\ R^2 z_2 u & -u^2 z_2 & -uvz_2 & z_2 u \\ R^2 z_2 v & -uvz_2 & -v^2 z_2 & z_2 v \\ R^2 z_3 & -uz_3 & -vz_3 & z_3 \end{pmatrix} \tag{4.9}$$

and $R^2=(u^2+v^2)/2$. Comparing (3.14)–(3.17) with (4.6)–(4.9) we see that the (p, u, v, ρ) system leads to a simpler preconditioner than does the (p, u, v, S) system. Choi and Merkle [4] pointed out that in the special case $\alpha=0$, that only the energy equation is modified, i.e., $z_2=0$ when $\alpha=0$. As before

$$(I+Q)^{-1}=I-\frac{\beta^2}{c^2}Q. \tag{4.10}$$

5. Compressible Isoenergetic System

In the two previous sections we have considered two possibilities for adding an additional differential equation to the incompressible equations. A different possibility is to use the fact that for the steady state Euler equations, when the flow originates from a common reservoir, the specific total enthalphy, $h=(E+p)/\rho$, is constant throughout the flow. Since we are only interested in steady state solutions, we can assumed that $h=h_0$ for all time. Such equations have been analyzed by Gottlieb and Gustaffson [7] and also Briley *et al.* [2], Viviand [20], and Rizzi and Eriksson [2]. Taking as our unknowns (p, u, v) the equations become in a general coordinate system (x, y),

$$\begin{aligned} \frac{J}{\rho c^2}p_t+\frac{1}{\rho c^2}(Up_x+Vp_y)+U_xY_y-v_xX_y-u_yY_x+v_yX_x&=0 \\ Ju_t+Uu_x+Vv_y+(p_xY_y-p_yY_x)/\rho&=0 \\ Jv_t+Uv_x+Vv_y+(-p_xX_y+p_yx_x)/\rho&=0, \end{aligned} \tag{5.1}$$

where

$$U=uY_y-vX_y, \qquad V=-uY_x+vX_x, \qquad J=X_XY_y-X_yY_x,$$

and

$$\rho=\left(\frac{\gamma}{\gamma-1}p\right)\Big/\left(h_0-\frac{u^2+v^2}{2}\right). \tag{5.2}$$

Using a preconditioning similar to that of the previous sections we consider

$$\begin{pmatrix} \frac{1}{\rho\beta^2} & 0 & 0 \\ \frac{\alpha u}{\rho\beta^2} & 1 & 0 \\ \frac{\alpha v}{\rho c^2} & 0 & 1 \end{pmatrix} \begin{pmatrix} p \\ u \\ v \end{pmatrix} + \begin{pmatrix} \frac{U}{\rho c^2} & Y_y & -X_y \\ Y_y/\rho & U & 0 \\ -X_y/\rho & 0 & U \end{pmatrix} \begin{pmatrix} p \\ u \\ v \end{pmatrix}_x$$

$$+ \begin{pmatrix} \frac{V}{\rho c^2} & -Y_x & X_x \\ -Y_x/\rho & V & 0 \\ X_x/\rho & 0 & V \end{pmatrix} \begin{pmatrix} p \\ u \\ v \end{pmatrix}_y = 0. \tag{5.3}$$

We see that the form of (5.3) is identical to that of (3.4). The only difference is that the coefficient ρ in (3.4) satisfies $\rho = \rho(p, S)$ while in (5.3) ρ is given by (5.2). However, the eigenvalue properties of the two systems are identical. In particular, it is evident from (5.3) that in the absence of preconditioning, i.e., $\beta = c$ and $\alpha = 0$, that (5.3) is simultaneously symmetrizable. In [7] there was an algebraic error and it was claimed that the isoenergetic system could not be symmetrized. In [19] we presented the matrices that would symmetrize the isoenergetic equations written in conservation variables. The proof of symmetry is more obvious when (p, u, v) variables are used as in (5.3). Furthermore, it follows from our previous results that (5.3) is symmetrizable for all α and β subject to the restraint (3.22).

We can rewrite (5.3) in terms of the conservation variables $(\rho, \rho u, \rho v)$ as

$$J \begin{pmatrix} z_1 p_t + \rho_t \\ u z_2 p_t + (\rho u)_t \\ v z_2 p_t + (\rho v)_t \end{pmatrix} + F_x + G_y = 0, \tag{5.4}$$

where

$$z_1 = \frac{\rho}{\beta^2 p}\left[c^2 - \beta^2 + \frac{\alpha(\gamma - 1)}{\gamma}(u^2 + v^2)\right]$$

and

$$z_2 = z_1 + \frac{\rho}{\beta^2 p}\frac{\alpha(\gamma - 1)}{\gamma}\left(h_0 - \frac{u^2 + v^2}{2}\right) = z_1 + \alpha/\beta^2. \tag{5.5}$$

It follows from (5.2) that

$$p_t = \frac{\gamma - 1}{\gamma}\left[\left(h_0 + \frac{u^2 + v^2}{2}\right)\rho_t - u(\rho u)_t - v(\rho v)_t\right]. \tag{5.6}$$

We can, therefore, rewrite (5.4) so that only time derivatives of ρ, ρu, ρv appear. We thus rewrite (5.4) as

$$J(I+Q)\begin{pmatrix}\rho \\ \rho u \\ \rho v\end{pmatrix}_t + F_x + G_y = 0$$

where I is the identity matrix and

$$Q = \frac{\gamma - 1}{\gamma}\begin{pmatrix} R^2 z_1 & -uz_1 & -vz_1 \\ uR^2 z_2 & -u^2 z_2 & -uvz_2 \\ vR^2 z_2 & -uvz_2 & -v^2 z_2 \end{pmatrix},$$

where $R^2 = h_0 + (u^2 + v^2)/2$. It also follows that

$$(I+Q)^{-1} = I - \frac{\beta^2}{c^2} Q. \tag{5.8}$$

As pointed out at the end of Section 3 we can, from (5.6), use p_t instead of ρ_t. Substituting into (5.4) we get a system which is similar to the incompressible equations. Now there is no difficulty near stagnation points.

6. Conclusion

In the previous sections, we have presented a unified theory of preconditioned methods for both incompressible and slow compressible flows. The pseudo-compressibility method for incompressible flow has been used by many authors, e.g., [3, 6, 15]. The original method and many examples are described in detail by Peyret and Taylor [10]. The work described here generalizes these previous works and hence there are many computational results to show the effectiveness of such an approach.

For the compressible equations at low Mach numbers several authors have also demonstrated the effectiveness of different preconditioners that correspond to the case $\alpha = 0$. Briley *et al.* [2] consider the isoenergetic equations. They present results using an implicit method for the Navier–Stokes equations with an algebraic turbulence model. Their preconditioning is covered by Section 5 of this study. Rizzi and Eriksson [12] based their preconditioning on a model of Viviand [20] for the inviscid isoenergetic equations. They present computational evidence of the usefulness of the preconditioning for the Euler equations in both two and three dimensions. They used an explicit three stage Runge-Kutta algorithm to obtain their solutions. Furthermore, Choi and Merkle [4] analyze a (p, u, v, ρ) formulation and present results for nozzle flow using an implicit ADI type algorithm. Their results are a subset of Section 4 with $\alpha = 0$. In [5] Merkle and Choi present an alternative approach to preconditioning that is effective for extremely small Mach numbers.

We thus see that many authors have successfully used variations of pseudo-compressibility preconditioning for both the incompressible and compressible equations. The various approaches used in these papers have been unified and generalized in this study.

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